A transverse condition of definable C^rG maps

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Abstract

Let G be a definably compact definable C^r group and $1 \le r < \infty$. Let X, Y be affine definable C^rG manifolds such that Y is definably compact and without boundary. We prove that for every definable C^rG map $f : X \to Y$ and for every definable C^rG submanifold Z of Y, there exists a definable $C^{r-1}G$ map $h : X \to Y$ such that h is definably $C^{r-1}G$ homotopic to f and h and $h|\partial X$ are transverse to Z.

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1 . Introduction.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$ and the term "definable" is used in the sense of "definable with parameters in $\mathcal{N}$" unless otherwise stated.

General references on o-minimal structures are [2], [3], also see [7].

A C^r manifold is a *definable C^r manifold* if it admits a finite system of charts whose gluing maps are of class definable C^r.

We say that a definable C^r manifold X is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f : (a, b) \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in X.

A definable C^r manifold G is a *definable C^r group* if G is a group and the group operations $G \times G \to G, G \to G$ are definable C^r maps. A definable C^r group G is *definably compact* if G is definably compact.

Let G be a definable C^r group. A *definable C^rG manifold* is a pair (X, ϕ) consisting of a definable C^r manifold X and a definable C^r action $\phi : G \times X \to X$ on X of G. For abbreviation, we write X instead of (X, ϕ).

Definable C^rG manifolds are studied in [4], [5].

Let G be a definably compact definable C^r group and $1 \le r < \infty$. A definable C^rG manifold is *affine* if it is definably C^rG diffeomorphic to a definable C^rG submanifold of some representation space of G. In this paper we are concerned with a transverse condition of definable C^rG maps between affine definable C^rG manifolds. In the rest of this paper, G means a definably compact definable C^r group and $1 \le r < \infty$ unless otherwise stated.

Let X, Y be definable C^r submanifolds of R^n and Z a definable C^r submanifold of Y. A definable C^r map $f : X \to Y$ *is* *transverse to* Z if for each $x \in X$ with $f(x) \in Z$, $(df)_x(T_xX) + T_{f(x)}Z = T_{f(x)}Y$.

Theorem 1.1. *Let X, Y, D be affine definable C^rG manifolds such that Y and D are without boundary, and $F : X \times D \to Y$ a definable C^rG map. Let Z be a definable C^rG submanifold of Y without boundary. If F and $F|\partial(X \times D)$ are transverse to Z, then for all $d \in D$ outside of a G invariant definable set of dimension $< \dim D$, f_d and $f_d|\partial X$ are transverse to Z, where $f_d : X \to Y$ is the map defined by $f_d(x) = F(x, d)$.*

Let X, Y be definable C^rG manifolds and $f, h : X \to Y$ definable C^rG maps. We say that f is *definably C^rG homotopic* to h if there exists a definable C^rG map $F : X \times [0,1]_R \to Y$ such that $f(x) = F(x, 0)$ and $h(x) = F(x, 1)$ for all $x \in X$, where the G action on $[0,1]_R = \{t \in R | 0 \leq t \leq 1\}$ is trivial.

Theorem 1.2. *Let X, Y be affine definable C^rG manifolds such that Y is definably compact and without boundary. Then for every definable C^rG map $f : X \to Y$ and for every definable C^rG submanifold Z of Y, there exists a definable $C^{r-1}G$ map $h : X \to Y$ such that h is definably $C^{r-1}G$ homotopic to f and h and $h|\partial X$ are transverse to Z.*

Theorem 1.2 is an equivariant version of 5.6 [1].

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. We have the following theorem when $\mathcal{N} = \mathcal{M}$.

Theorem 1.3. *Let $\mathcal{N} = \mathcal{M}$ and G a compact definable C^r group. Let X, Y be affine definable C^rG manifolds such that Y is compact and without boundary. Let $f : X \to Y$ be a definable C^rG map and Z a definable C^rG submanifold of Y without boundary. If $f|\partial X$ is transverse to Y and ∂X is compact, then there exists a definable $C^{r-1}G$ map $h : X \to Y$ such that h is definably $C^{r-1}G$ homotopic to f, $f = h$ on ∂X and h is transverse to Z.*

2. Proof of Theorem 1.1 and Theorem 1.2.

Let G be a definably compact definable C^r group. A group homomorphism from G to some $O_n(R)$ is a *representation* if it is a definable C^r map, where $O_n(R)$ means the nth orthogonal group of R. A *representation space* Ω of G is R^n with the orthogonal action induced from a representation of G. A *definable C^rG submanifold* of Ω of G is a G invariant definable C^r submanifold of Ω. A *definable G set* means a G invariant definable subset of some representation space of G.

Let Y be a definable C^rG submanifold of an l-dimensional representation space Ω of G. For any $y \in Y$, let $N_y(Y)$ be the orthogonal complement of the tangent space $T_y(Y)$ of Y at y with respect to the usual inner product on Ω. We define the *normal bundle* $N(Y) \subset R^{2l}$ as the union $\cup_{y \in Y}\{y\} \times N_y(Y)$. As in [6], the above R^{2l} is a representation space Ξ of G such that $\Omega \times \{0\}$ is G invariant.

Proposition 2.1. *Let Y be a definable C^rG submanifold without boundary of an l-dimensional representation space Ω of G. Then $N(Y)$ is an l-dimensional definable $C^{r-1}G$ submanifold of Ξ and the projection $\pi : N(Y) \to Y$, sending $(y, v) \in \{y\} \times N_y(Y)$ to y, is a submersive definable $C^{r-1}G$ map.*

Proof. By 4.1 [1], $N(Y)$ is a definable C^{r-1} submanifold of R^{2l} and π is a submersive definable C^{r-1} map. By the construction of Ξ and $N(Y)$, $N(Y)$ is G invariant and π is a G map. □

Recall that a definable subset of R^n is definably compact if and only if it is bounded and closed. For a G invariant definably compact definable subset Z of a G invariant definable subset X of a representation space Ω of G and for an $\epsilon > 0$, the *G invariant ϵ-neighborhood of Z in X* is the set of all points of X whose distance from some point of Z less than ϵ. Note that this definable set is G invariant because the action of G is orthogonal.

Proposition 2.2. *Let X be a G invariant definable subset of a representation space Ω of G and Z a G invariant definably compact definable subset of X. For any G invariant definable open subset U of X containing Z, there exists some $\epsilon > 0$ such that U contains the G invariant ϵ-neighborhood W of Z in X.*

Proof. By 4.2 [1], there exists some $\epsilon > 0$ such that U contains the ϵ-neighborhood W of Z in X. By the construction and the action on Ω is orthogonal, W is G invariant. □

Lemma 2.3. *Let X, Y be definable C^rG submanifolds without boundary of a representation spaces Ω, Ξ of G, respectively, and $f : X \to Y$ a definable C^rG map. Suppose that f maps a definably compact C^rG submanifold Z of X definably C^rG diffeomorphically onto $f(Z)$ and that $(df)_x : T_xX \to T_{f(x)}Y$ is an isomorphism for each $x \in Z$. Then there exist G invariant definable open neighborhoods U (resp. V) of Z (resp. $f(Z)$) in X (resp. Y) such that $f|U : U \to V$ is a definable C^rG diffeomorphism.*

Proof. By 4.3 [1], there exist definable open neighborhoods U (resp. V) of Z (resp. $f(Z)$) in X (resp. Y) such that $f|U : U \to V$ is a definable C^r diffeomorphism. By the construction of them and the action on Ω is orthogonal, U, V are G invariant and f is a definable C^rG diffeomorphism. □

Theorem 2.4. *Let Y be a definably compact definable C^rG submanifold of a representation space Ω of G without boundary. Then there exist a G invariant definable open neighborhood V of X in Ω and a definable $C^{r-1}G$ submersion $\theta : V \to X$ such that $\theta|X = id_X$.*

If $\mathcal{N} = \mathcal{M}$, then Theorem 2.4 is proved in 1.2 [4] when $1 \le r < \infty$ and in 2.24 [5] when $r = \infty$.

Proof of Theorem 2.4. Define a definable $C^{r-1}G$ map $h : N(X) \to \Omega$ by $h(x, v) = x + v$. Then $h|X \times \{0\}$ is a definable $C^{r-1}G$ map onto X and $(dh)_{(x,0)}$ is an isomorphism for each $x \in X$. By Lemma 2.3, there exist G invariant definable open neighborhoods U (resp. V) of $X \times \{0\}$ (resp. X) in $N(X)$ (resp. Ω) such that $h : U \to V$ is a definable $C^{r-1}G$ diffeomorphism. Therefore $\pi \circ h^{-1} : V \to X$ is the required submersion, where $\pi : N(X) \to X$ denotes the projection. □

Proposition 2.5. *Let X (resp. Y) be a definable C^rG submanifold of a representation space Ω (resp. Ξ) such that Y is definably compact and without boundary, and $f : X \to Y$ a definable C^rG map. Let D be the open unit disk of Ξ. Then there exists a definable $C^{r-1}G$ map $F : X \times D \to Y$ such that $F(x, 0) = f(x)$ and for fixed $x \in X$ the map $F_x : D \to Y$ defined by $F_x(d) = F(x, d)$ is a submersive definable $C^{r-1}G$ map.*

Proof. By Theorem 2.4, there exist a G invariant definable open neighborhood U of Y in Ξ and a definable $C^{r-1}G$ submersion $\theta : U \to Y$. By Proposition 2.2, we may assume that U is an ϵ-neighborhood. Thus $F : X \times D \to Y, F(x, d) = \theta(f(x) + \epsilon d)$ is the required map. □

The following theorem is a definable version of Sard's theorem.

Theorem 2.6 ([1])**.** *Let $X \subset R^n, Y \subset R^m$ be definable C^1 manifolds and $f : X \to Y$ a definable C^1 map. Then the set of critical values of f has dimension of less than $\dim Y$.*

Proof of Theorem 1.1. Since X, Y, D are affine, we may assume that they are definable C^rG submanifolds of a representation space of G.

Let $\pi : X \times D \to D$ be the projection and d a regular value of $\pi|F^{-1}(Z)$ and $\pi|\partial(F^{-1}(Z))$.

Since d is a regular value of $\pi|F^{-1}(Z)$ and $(df_d)_x(T_xX) = (dF)_{(x,d)}(T_x(X \times \{d\}))$, $T_x(X \times \{d\})$ is a direct sum of $T_{(x,d)}(F^{-1}(Z))$ in $T_{(x,d)}(X \times D)$. Thus its image of $(dF)_{(x,d)}$ is a direct sum of T_zZ in T_zY. Hence for any $x \in X$ with $f_d(x) \in Z$, $(df_d)_x(T_xX) + T_zZ = T_zY$. By Theorem 2.6, we have Theorem 1.1.

The above argument works for $\pi|\partial(F^{-1}(Z))$. $\square$

Proof of Theorem 1.2. By Proposition 2.5, there exists a definable C^rG map $F : X \times D \to Y$ such that F and $F|\partial(X \times D)$ are transverse to Z. By Theorem 1.1, there exists a point d in D such that $f_d : X \to Y, f_d(x) = F(x, d)$ is transverse to Z and $f_d|\partial X$ is transverse to Z. By construction of f_d, f_d is definably $C^{r-1}G$ homotopic to f. $\square$

Lemma 2.7 (5.7 [1])**.** *Let A be a definable closed subset of R^n. Then there exists a definable C^r function $f : R^n \to R$ such that $A = f^{-1}(0)$.*

We have the following equivariant version of Lemma 2.7 when $\mathcal{N} = \mathcal{M}$.

Theorem 2.8. *Let $\mathcal{N} = \mathcal{M}$, G a compact definable C^r group and A a G invariant definable closed subset of a representation space Ω of G. Then there exists a G invariant definable C^r function $f : \Omega \to \mathbb{R}$ such that $A = f^{-1}(0)$.*

Proof. Since G is a compact definable C^r group, the orbit map $\pi : \Omega \to \Omega/G \subset \mathbb{R}^l$ is a G invariant proper polynomial map. Thus $\pi(A)$ is closed in $\mathbb{R}^l$. By Lemma 2.7, there exists a deinable C^r function $\phi : \mathbb{R}^l \to \mathbb{R}$ such that $\pi(A) = \phi^{-1}(0)$. Therefore $f := \pi \circ \phi : \Omega \to \mathbb{R}$ is the required function. $\square$

Proof of Theorem 1.3. By Theorem 2.8, there exists a G invariant definable C^r function $\phi : X \to [0, 1]$ such that $\partial X = \phi^{-1}(0)$. By Proposition 2.5, we have a definable $C^{r-1}G$ map $F : X \times D \to Y$ such that $F(x, 0) = f(x)$ and for fixed $x \in X$ the map $F_x : D \to Y$ defined by $F_x(d) = F(x, d)$ is a submersive definable $C^{r-1}G$ map. Define $H : X \times D \to Y, H(x, d) = F(x, \phi^2(x)d)$.

We claim that H and $H|\partial(X \times D)$ are transverse to Z. Let $x \in X$ such that $\phi(x) \neq 0$. Then the map $d \mapsto H(x, d)$ is the composition of the two submersions $D \to D, d \mapsto \phi^2(x)d$ and $D \to Y, d \mapsto F(x, d)$. If $\phi(x) = 0$, then $x \in \partial X$. By the chain rule, $(dH)_{(x,d)} : T_xX \times T_dD \to T_{H(x,d)}Y$ is expressed by $(dH)_{(x,d)}(v, w) = (dF)_{(x,\phi^2(x)d)}(v, \phi^2(x)\cdot w + 2\phi(x)(d\phi)_x(v) \cdot d) = (dF)_{(x,0)}(v, 0) = (df)_x(v)$. Thus the image of $(dH)_{(x,d)}$ coincides that of $(df)_x$. Hence the claim is proved.

By Theorem 1.2, there exists $d \in D$ such that $h(x) := H(x, d)$ is transverse to Z, and h is the required map. $\square$

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